Syllabus

(1) Analysis of Stress and Strain.

(2) Hooke's Law.

(3) Strain-energy function.

(4) Equations of elasticity and uniqueness of solution.

(5) Problems of plane stress and strain.

(6) Extension, torsion and flexure of homogeneous beams.

(7) St. Venant's Problem

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Lecture Notes on Elasticity during 1962–63

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(Following Sokolnikoff).

Lecture 1 – 19/7/62 – Defns of elastic body, stress and strain – To deal with stress first – Body forces and surface forces at any point of the body – Notations (1, 2, 3) & (x, y, z), and mixture of both – Body force: $R_i = \int_\tau F_i \, d\tau$ – Surface force: $\vec{T}_i$ (not along $\vec{\nu}$ as in fluid mechanics); resolved parts along $\vec{\nu}$ and $\perp$ to $\vec{\nu}$ ie tension ~~and~~ or compression, and shear – Used of Π^d picture to obtain $\overset{j}{T}_i = \tau_{ji}$ (stress tensor); tension or compression – Resolution of body forces and surfaces using Π^d picture to obtain $\vec{T}_i = \tau_{ji} \nu_j$ – Derivation of equilibrium for the whole body, $\tau_{ji,j} = -F_i$.

Lecture 2 – 21/7/62 – Putting sum of moments equal to zero leading to $\tau_{ji} = \tau_{ij}$ – Reciprocity theorem $\vec{T}' \cdot \vec{\nu} = \vec{T} \cdot \vec{\nu}'$ – Proof tensor character of τ_{ij} – Invariance of τ_{ii} – Exx. 1, 2, 3 on p. 44 solved (Solns on p. 1 of 1961–62 Notes).

Lecture 3 – 28/7/62 – Ex. 4. p. 44 solved (Soln in above) – Stress quadric of Cauchy – Tensor rule for τ_{ij} from invariance of $\tau_{ij} x_i x_j$ for change of axes – Normal at point of quadric $\parallel^l$ to T and hence construction of T given N – Principal axes and directions of stress; reality of roots and orthogonality of directions – Invariants $\Theta_1, \Theta_2, \Theta_3$ in terms of τ_1, τ_2, τ_3 only.

Lecture 4 – 2/8/62 – $\Theta_1, \Theta_2, \Theta_3$ in terms of τ_{ij} by expanding $|\tau_{ij} - \delta_{ij}\tau| = 0$, and comparing coefficients of τ^2, τ, and constant term in $-\tau^3 + \Theta_1\tau^2 - \Theta_2\tau + \Theta_3 = 0$ – Transformation to principal axes using $\tau_{ij} = l_{i\alpha} l_{j\alpha} \tau_\alpha$ – Several cases of stress-quadric (i) Ellipsoid (using $N = K^2/A^2$) gives tensile stress with $+K^2$, and compressive stress with $-K^2$, (ii) $\tau_1 > 0, \tau_2 > 0, \tau_3 < 0$, $+K^2$ gives hyperboloid of one sheet, and $-K^2$ of two sheets, and if normal to the element meets the first stress is tensile, and if the second stress is compressive; (iii) $\tau_1 < 0, \tau_2 < 0, \tau_3 > 0$, same as (ii) but only different regions, (iv) Sphere – Stress purely normal whatever the orientation of surface element – Theorem on maximum shearing stress.

5th Lecture – 4/8/62 – Discussion of the Mohr Circle diagram.

6th Lecture (9/8/62) – Problem on p. 53 worked out (Soln. on p. 5 of last year's Lecture Notes) – Discussion of the maximum normal and shear stresses without the use of the Mohr diagram (as in revising p. 3 of last year's Lecture notes)

7th Lecture – 16/8/62 – Special examples of stress: (a) Purely normal stress, (b) Simple tension or compression, (c) Shearing stress

8th Lecture – 18/8/62 – The 5th special case (d) of plane stress – Finding directions of principal axes,

(iii) ϵ_{44} real & is equal to zero, hence both L.H.S & R.H.S of (B') = 0

(iv) ϵ_{kl} real (we need not consider case of $k = l$, for then $\epsilon_{kl} = 0$ & just as in (iii) (B') holds.

For $k \neq l$, (B') becomes $\epsilon^{c}_{kl} \gamma_k \gamma_l = + \epsilon_{kl} \gamma_4 \gamma_k \gamma_l \gamma_4$

$L.H.S = \epsilon^{a}_{kl} \gamma_k \gamma_l = + \epsilon_{kl} \gamma_k \gamma_l$; $R.H.S = + \epsilon_{kl} \gamma_4^2 \gamma_k \gamma_l = + \epsilon_{kl} \gamma_k \gamma_l$ & hence (B') holds.

(4) (iv) $K_\mu = i \bar{\psi} \gamma_5 \gamma_\mu \psi$ - Here $\gamma_5 \gamma_\mu$ has only 4 values, ~~for eg~~ $\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_\mu$ ($\mu = 1, 2, 3, 4$) ie $-\gamma_2 \gamma_3 \gamma_4$, $-\gamma_3 \gamma_1 \gamma_4$, $-\gamma_1 \gamma_2 \gamma_4$ and $\gamma_1 \gamma_2 \gamma_3$ & K_μ is called a pseudo-vector or a vector like A.M for which the law of transformation

under L.T's is given by $K_\mu' = |a_{\alpha\beta}| a_{\mu\nu} K_\mu$ ie $|\ |$ can be ± 1

$$K_\mu' = i \bar{\psi}' \gamma_5 \gamma_\mu \psi' = i \bar{\psi} S^{-1} \gamma_5 \gamma_\mu S \bar{\psi}$$

$$= i \bar{\psi} (S^{-1} \gamma_1 S)(S^{-1} \gamma_2 S)(S^{-1} \gamma_3 S)(S^{-1} \gamma_4 S) S^{-1} \gamma_\mu S \bar{\psi}$$

$$= a_{1\lambda} a_{2\mu} a_{3\nu} a_{4\sigma}\, i \bar{\psi} \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\sigma S^{-1} \gamma_\mu S \psi$$

$$= a_{1\lambda} a_{2\mu} a_{3\nu} a_{4\sigma} \epsilon_{\lambda\mu\nu\sigma} \cdot i \bar{\psi} \gamma_1 \gamma_2 \gamma_3 \gamma_4 S^{-1} \gamma_\mu S \psi$$

$$= |a_{\mu\nu}| \cdot i \bar{\psi} \gamma_5 \gamma_\mu S^{-1} \gamma_\mu S \psi$$

$$= |a_{\mu\nu}| \, a_{\mu\nu} \cdot i \bar{\psi} \gamma_5 \gamma_\nu \psi$$

$$= |a_{\mu\nu}| \cdot a_{\mu\nu} K_\nu .$$

(v) $N = i \bar{\psi} \gamma_5 \psi$ - Pseudo-scalar ie $N' = |a_{\mu\nu}| N$. This is ~~easily~~ proved as in Mandl, p. 221

Michell - Kolosov relation in the genral case, and its modification for this particular transformation (as on p. 6 of last year's notes) – Summary of results of stress under six headings – Introduction to strain; Rigid & elastic bodies – affine transformations & their properties; infinitesimal affine transformations, and their superposition.

9th Lecture – 25/8/62 – Genral displacement of vector: $\delta A_i = \alpha_{ij} A_j$ including both rigid body displacements and elastic deformations – Division into symmetric & antisymmetric part of α_{ij} (e_{ij} & ω_{ij}) with e_{ij} corresponding to elastic deformations – Geometrical interpretation of the components e_{ii} and e_{ij} of the strain tensor ($\delta A_i = e_{ij} A_j$) with e_{11} and e_{23} as particular cases.

10th Lecture – 30/8/62 – Strain quadric – change of axes & tensor character of e_{ij} – $\delta\vec{A}$ along the normal at P to the strain quadric – Principal axes of strain given by $\| e_{ij} - e\delta_{ij} \| = 0$; roots of this being real – Invariants $\vartheta_1, \vartheta_2, \vartheta_3$ in terms of e_1, e_2, e_3 and e_{ij} – Interpretation of $\vartheta_1 = \vartheta = e_1 + e_2 + e_3$ as the cubical dilatation.

11th Lecture – 1/9/62 – Case of genral infinitesimal deformation viz $\delta A_i = u_{i,j} A_j$ ($u_{i,j} = \partial u_i / \partial x_j$) (ie unlike the affine case $u_{i,j}$ varies from point to point, while the α_{ij} is constant throughout; but in the neighbourhood of a point, the trn is affine) – Introduction ω_{ij} & e_{ij} by $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$ – Dilation $\vartheta = u_{i,i}$ varies from point to point – Examples of strain: Four cases: (a) Uniform dilatation, (b) Simple extension in axes (l_1, l_2, l_3) (c) Shearing strain equivalent to equal extension & contraction of two orthogonal linear elements; (d) Plane strain – Strain quadrics in the four cases.

12th Lecture – 6/9/62 – Examples on p. 24 of Sokolnikoff – Deduction of St. Venant's compatibility conditions as necessary conditions following the method of Ex 6, p. 33 of Sokolnikoff and showing partly that the 81 conditions really reduce to only 6.

13th Lecture – 15/9/62 – Revision of the reduction of the 81 conditions to only 6:

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \quad -- \quad (A)$$

for the case $ij = 11$, reduces to an identity except when $kl = 23$, & similarly for $ij = 22$, $kl = 31$, $ij = 33$, $kl = 12$

Again for $ij = 23$, " $kl = 23$ & $ij = 31$, $kl = 31$, $ij = 12$, $kl = 12$.

thus giving only 6 conditions written out in the usual form.

14th Lecture – 20/9/62 – Suffic (A) also proved to be sufficient conditions as in Sokolnikoff, p. 27, only with this difference* that U_{jl} was taken $= e_{jl} - x_k (e_{lj,K} - e_{kl,j})$ without introducing the x_K' which appears unnecessary – Mentioned that the displacements given by $u_i = u_j^0 + (x_K - x_K^0)\omega_{jK}^0 + \int_{P_0}^{P} U_{jl}\, dx_l$ can be shown to satisfy the differential equation $u_{i,j} + u_{j,i} = 2e_{ij}$ – Postponed This to the next lecture.

vectors are defined by the transformation (under L·T) by $K_\mu' = |a_{\alpha\beta}| a_{\mu\nu} K_\nu$. Here

$$K_\mu' = i\bar{\psi}' \gamma_5 \gamma_\mu \psi' = i \bar{\psi} S^{-1} \gamma_5 \gamma_\mu S \psi$$

74* Lecture – (3) of last lecture, and the remaining four components of (4) of last lecture.

(24|2|65) (3) Since the proper L·T's form a group, any element of the group can be considered to be formed out of infinitesimal elements, it is sufficient to prove the existence of S satisfying (A) & (B) for infinitesimal L·T's. Let such a one be specified by $a_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu}$ (squares of ϵ neglected);

then $a_{\mu\alpha} a_{\mu\beta} = \delta_{\alpha\beta}$ requires $(\delta_{\mu\alpha} + \epsilon_{\mu\alpha})(\delta_{\mu\beta} + \epsilon_{\mu\beta}) = \delta_{\mu\alpha}\delta_{\mu\beta} + \delta_{\mu\alpha}\epsilon_{\mu\beta} + \delta_{\mu\beta}\epsilon_{\mu\beta} = \delta_{\alpha\beta}$

i.e. since $\delta_{\mu\alpha}\delta_{\mu\beta} = \delta_{\alpha\beta}$, we should have $\delta_{\mu\alpha}\epsilon_{\mu\beta} + \delta_{\mu\beta}\epsilon_{\mu\alpha} = 0$ or $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$

This is what characterises an infinitesimal L·T: $x_\mu' = x_\mu + \epsilon_{\mu\nu} x_\nu$ ($\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$).

Let $S = I + \frac{1}{2}\epsilon_{\mu\nu} T^{\mu\nu}$, where $T^{\mu\nu} = -T_{\mu\nu}$; then S = unit matrix, and $S^{-1} = I - \frac{1}{2}\epsilon_{\mu\nu} T^{\mu\nu}$

If S should satisfy (A) i.e. $S^{-1}\gamma_\lambda S = a_{\lambda\mu}\gamma_\mu$, we should have

$$(I - \tfrac{1}{2}\epsilon_{\mu\nu} T^{\mu\nu})\gamma_\lambda (I + \tfrac{1}{2}\epsilon_{\mu\nu} T^{\mu\nu}) = a_{\lambda\mu}\gamma_\mu = (\delta_{\lambda\mu} + \epsilon_{\lambda\mu})\gamma_\mu = \gamma_\lambda + \epsilon_{\lambda\mu}\gamma_\mu$$

$$\gamma_\lambda + \tfrac{1}{2}\gamma_\lambda \epsilon_{\mu\nu} T^{\mu\nu} - \tfrac{1}{2}\epsilon_{\mu\nu} T^{\mu\nu}\gamma_\lambda = \gamma_\lambda + \epsilon_{\lambda\mu}\gamma_\mu$$

i.e. $\frac{1}{2}(\gamma_\lambda \epsilon_{\mu\nu} T^{\mu\nu} - \epsilon_{\mu\nu} T^{\mu\nu}\gamma_\lambda) = \epsilon_{\lambda\mu}\gamma_\mu = \frac{1}{2}\epsilon_{\mu\nu}(\delta_{\mu\lambda}\gamma_\nu - \delta_{\nu\lambda}\gamma_\mu)$, as can be easily verified

i.e. this is satisfied for any $\epsilon_{\mu\nu}$ if

$$\gamma_\lambda T^{\mu\nu} - T^{\mu\nu}\gamma_\lambda = \delta_{\mu\lambda}\gamma_\nu - \delta_{\nu\lambda}\gamma_\mu \quad \text{---(1)}$$

This eqn enables determination of the $T^{\mu\nu}$ in terms of the γ-matrices & hence determination of S, thus proving the existence of S satisfying (A). A solution of (1) of the $T^{\mu\nu}$ skew-symmetric in μ,ν is given ($\mu \neq \nu$)

by $T^{\mu\nu} = \frac{1}{2}\gamma_\mu\gamma_\nu$ ..... (2)

as can be shown by considering cases (i) $\lambda \neq \mu$ or ν, (ii) $\lambda = \mu \neq \nu$; (iii) $\lambda = \nu \neq \mu$.

We have next to show that (1) satisfies (B) also, and it is enough to show that (2) satisfies (B)

(B) is $\gamma_4 S^* \gamma_4 = S^{-1}$.

$S^* = 1 + \frac{1}{2} T^{*\mu\nu} \epsilon^*_{\mu\nu}$; $S^{-1} = 1 - \frac{1}{2}\epsilon_{\mu\nu} T_{\mu\nu}$

and (B) becomes $\gamma_4 (1 + \frac{1}{2} T^{*\mu\nu} \epsilon^*_{\mu\nu}) \gamma_4 = 1 - \frac{1}{2}\epsilon_{\mu\nu} T_{\mu\nu}$

i.e. using $\gamma_4^2 = 1$, $\gamma_4 T^*_{\mu\nu} \epsilon^*_{\mu\nu} \gamma_4 = -\epsilon_{\mu\nu} T_{\mu\nu}$

and $T^*_{\mu\nu} = \frac{1}{2}\gamma_\nu^* \gamma_\mu^* = \frac{1}{2}\gamma_\nu\gamma_\mu = -\frac{1}{2}\gamma_\mu\gamma_\nu$. Hence (B) becomes

$$\gamma_4\gamma_\mu\gamma_\nu \epsilon^*_{\mu\nu} \gamma_4 = -\epsilon_{\mu\nu}\gamma_\mu\gamma_\nu \text{ or } \gamma_\mu\gamma_\nu \epsilon^*_{\mu\nu} = -\epsilon_{\mu\nu}\gamma_4\gamma_\mu\gamma_\nu\gamma_4$$

(+) or $\epsilon^*_{\mu\nu}\gamma_\mu\gamma_\nu = +\epsilon_{\mu\nu}\gamma_4\gamma_\mu\gamma_\nu\gamma_4$ ~~(B)~~

$\epsilon_{\mu\nu}$ is not treated as a matrix but as a coefficient $\therefore \epsilon^*_{\mu\nu} = \epsilon^c_{\mu\nu}$ → or $\epsilon^c_{\mu\nu}\gamma_\mu\gamma_\nu = +\epsilon_{\mu\nu}\gamma_4\gamma_\mu\gamma_\nu\gamma_4$ -- (B') (where ϵ^c is complex conjugate of ϵ)

From the nature of the definition ϵ_{k4}, ϵ_{44} are imaginary & ϵ_{44} & ϵ_{kl} are real, and we can verify (B') for these

several cases (i) $\epsilon^c_{k4}\gamma_k\gamma_4 = +\epsilon_{k4}\gamma_4\gamma_k\gamma_4\gamma_4 = -\epsilon_{k4}\gamma_k\gamma_4$

5th Lecture – 29/9/62 – Hooke's law – Elastic limit and proportional limit – yield point – $T = f(e)$ for elastic & plastic materials – Generalised Hooke's law $\tau_{ij} = c_{ijkl}\, e_{kl}$ with symmetry of c_{ijkl} and i, j and k, l – $\tau_i = c_{ij}\, e_j$ $(i, j = 1, \ldots 6)$ & $e_i = c_{ij}\, \tau_j$ – Reduction from 36 to 21 constants when the strain-energy density fn $W = \frac{1}{2} c_{ij}\, e_i\, e_j$, such that $\tau_i = \frac{\partial W}{\partial e_i}$, exists

6th Lecture – 4/10/62 – Elastic and geometric symmetry – (i) Elastic symmetry w.r.t. a plane, (ii) w.r.t three mutually $\perp^r$ planes – Elastic symmetry determined from invariance of c_{ij} ie $\tau_i' = c_{ij}\, e_j'$ and $\tau_i = c_{ij}\, e_j$ when transformation of coordinates is made – ~~Matrix for the case of homogeneous isotropic solid~~

7th Lecture – 6/10/62 – Matrix c_{ij} for the case of homogeneous isotropic solid –

8th Lecture (After vacation) – 17/11/62 – Did no work.

9th Lecture – 22/11/62 – Worked out Q.7. given in the terminal examination about maximum shearing stress without using Mohr's diagram & adopting the second method in my last year's notes viz $S^2 = \nu_2^2 \nu_3^2 (\tau_2 - \tau_3)^2 + \nu_3^2 \nu_1^2 (\tau_3 - \tau_1)^2 + \nu_1^2 \nu_2^2 (\tau_1 - \tau_2)^2$ – Also worked out ~~in a~~ the case $\tau_2 = \tau_3$.

26th Lecture – 6/12/62 – Homogeneous isotropic media – Derivation of $\tau_{ij} = \lambda \delta_{ij} \vartheta + 2\mu e_{ij}$ – $\Theta = (3\lambda + 2\mu)\vartheta$

$$e_{ij} = -\frac{\lambda \delta_{ij}}{2\mu(3\lambda + 2\mu)} \Theta + \frac{1}{2\mu} \tau_{ij}$$

(λ & μ called Lame's consts.) – Principal axes of strain (ie $e_{23} = e_{31} = e_{12} = 0$) coincide with pr. axes of stress – Case of simple tension with $\tau_{11} = T$, all the $\tau_{ij} = 0$, introduction of σ and E – Case of simple shear; $\mu = T/\alpha_{23}$ – From $\mu = E/2(1+\sigma)$ $\mu > 0$ since $E, \sigma > 0$ //.

27th Lecture (After Christmas Vacation) – 10/1/63 – Case of hydrostatic pressure of uniform intensity p on or over surface of the body ie $\tau_{ij} = -p\delta_{ij}$. Both $\vec{T}_i = \tau_{ij} \nu_j$ and $\tau_{ij,j} = 0$ for surface & interior eqns are satisfied – $\Theta = -3p$, and from $e_{ij} = -\frac{\lambda \delta_{ij}}{2\mu(3\lambda + 2\mu)} \Theta + \frac{\tau_{ij}}{2\mu}$ we get

$e_{11} = e_{22} = e_{33} = -\frac{p}{3\lambda + 2\mu}$, $e_{23} = e_{31} = e_{12} = 0$.

satisfying compatibility eqns – $\vartheta = e_{11} + e_{22} + e_{33} = -p/(\lambda + \frac{2}{3}\mu)$ or $\vartheta = -p/R$ or $k = -p/\vartheta$ with $R = \lambda + \frac{2}{3}\mu$ and R = compressive stress / cub. comp? = modulus of compression > 0. – Derivation of $R = E/3(1-2\sigma)$ & since $R > 0$, $\sigma < 1/2$ & from $\lambda = E\sigma/(1+\sigma)(1-2\sigma)$ it follows that $\lambda > 0$ –

Compn $e_{ij} = \frac{1+\sigma}{E} \tau_{ij} - \frac{\sigma}{E} \delta_{ij} \Theta$,

p. 71. Ex. 2 – From above eqn $\vartheta = \frac{1+\sigma}{E}\Theta - \frac{3\sigma}{E}\Theta = \frac{1-2\sigma}{E}\Theta = \frac{\Theta}{3R}$ ie $\Theta = 3K\vartheta$

Ex. 4 – Did the first part viz. eqn for λ.

– Proper L·T's and forming a group – (Proof of a_{kl}, a_{ik}, a_{44} being real & a_{k4}, a_{4k} being imaging)

73rd Lecture – (1) Proper L·T's forming a group – (2) Correct derivation of $S^{-1}\gamma_\lambda S = a_{\lambda\mu}\gamma_\mu$ (A) (Correcting the mistake in the last step yesterday) – Derivation of $\gamma_4 S^*\gamma_4 = S^{-1}$ (B) – (3) Proof of existence of an S for every L·T. satisfying (A) and (B) – (4) Covariants of the five types constructed out of the ψ, $\bar\psi$ and the γ_μ.

(23/2/65)

(1) [To prove the identity exists for L·T's, consider $x_\mu' = \delta_{\mu\nu}x_\nu$; this an L·T since $\delta_{\mu\lambda}\delta_{\lambda\rho} = \delta_{\mu\rho}$ as can be easily verified. That inverse exists has been proved. Re. product of two L·T's being an L·T. see last years lecture notes. Also proof that c_{kl}, c_{44} are real & c_{4k}, c_{k4} are imaging follows from $c_{\mu\nu} = b_{\mu\lambda}a_{\lambda\nu}$ & also $|c_{\mu\nu}| = |b_{\mu\lambda}||a_{\lambda\nu}| = 1$]

(2) [(A) In $\gamma_\mu\frac{\partial\psi}{\partial x_\mu} + M\psi = 0$, write $\psi = S^{-1}\psi'$ and $\frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x_\lambda'}\cdot\frac{\partial x_\lambda'}{\partial x_\mu} = \frac{\partial}{\partial x_\lambda'}a_{\lambda\mu}$, getting

$\gamma_\mu\frac{\partial}{\partial x_\lambda'}a_{\lambda\mu}S^{-1}\psi' + MS^{-1}\psi' = 0$ ie $a_{\lambda\mu}\gamma_\mu S^{-1}\frac{\partial\psi'}{\partial x_\lambda'} + MS^{-1}\psi' = 0$ & multiplying on the left by S, $a_{\lambda\mu}S\gamma_\mu S^{-1}\frac{\partial\psi'}{\partial x_\lambda'} + M\psi' = 0$. Hence comparing with $\gamma_\lambda\frac{\partial\psi'}{\partial x_\lambda'} + M\psi' = 0$, we get

$a_{\lambda\mu}S\gamma_\mu S^{-1} = \gamma_\lambda$ and again multiplying on the left by S^{-1} and on the right by S, we get $a_{\lambda\mu}\gamma_\mu = S^{-1}\gamma_\lambda S$, which is (A).

[(B), ~~Using the condition that the continuity eqn $\frac{\partial s_\mu}{\partial x_\mu} = 0$ should remain invariant requires~~

~~$\frac{\partial s_\mu}{\partial x_\mu} = \frac{\partial s_\mu'}{\partial x_\mu'}\cdot$ Putting $\frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x_\lambda'}a_{\lambda\mu}$ and $s_\mu = i\bar\psi\gamma_\mu\psi$ $(\bar\psi = \psi^*\gamma_4)$~~

ie ~~i and $\psi = S^{-1}\psi'$, $\bar\psi = \psi^*S^{-1}$ ie $\bar\psi' = \psi^*S^*\gamma_4\psi'^* =$~~

Using the condition that s_μ should be a four-vector in both systems requires that s_μ should transform like x_μ ie $s_\mu' = a_{\mu\nu}s_\nu$ (†) & using definition of $s_\mu = i\bar\psi\gamma_\mu\psi$ (where $\bar\psi = \psi^*\gamma_4$), we should have $\bar\psi'\gamma_\mu\psi' = a_{\mu\nu}\bar\psi\gamma_\nu\psi$, and using $\gamma_\mu = a_{\mu\nu}S\gamma_\nu S^{-1}$ this becomes $\bar\psi' a_{\mu\nu}\bar\psi' S\gamma_\nu S^{-1}\psi' = a_{\mu\nu}\bar\psi\gamma_\nu\psi$ or $a_{\mu\nu}\bar\psi' S\gamma_\nu\psi = a_{\mu\nu}\bar\psi\gamma_\nu\psi$

ie $\bar\psi' S = \bar\psi$ or $\bar\psi' = \bar\psi S^{-1}$.

(†) Alternatively if $\frac{\partial s_\lambda'}{\partial x_\lambda'} = \frac{\partial s_\mu}{\partial x_\mu}$, $\frac{\partial s_\lambda'}{\partial x_\lambda'} = \frac{\partial s_\mu}{\partial x_\nu'}a_{\lambda\mu}$, $s_\lambda' = a_{\lambda\mu}s_\mu$, $\forall\ s_\mu' = a_{\mu\nu}s_\nu$

Using $\bar\psi' = \psi'^*\gamma_4$ and $\psi'^* = (S\psi)^* = \psi^*S^*$ ie $\bar\psi' = \psi^*S^*\gamma_4 = \bar\psi\gamma_4 S^*\gamma_4$ ($\because \bar\psi = \psi^*\gamma_4$ & $\rightarrow \psi^* = \bar\psi\gamma_4$, $\gamma_4^2 = 1$)

hence $\gamma_4 S^*\gamma_4 = S^{-1}$ which is (B)]

(3) [To be supplied from Pauli's article in Handbuch d. Physik.] ✗

(4) [(i) $I = \bar\psi\psi$; $\bar\psi'\psi' = \bar\psi S^{-1}S\psi = \bar\psi\psi$ ie I is an invariant

(ii) $s_\mu = i\bar\psi\gamma_\mu\psi$; $s_\mu' = i\bar\psi'\gamma_\mu\psi' = i\bar\psi S^{-1}\gamma_\mu S\psi = i\bar\psi a_{\mu\nu}\gamma_\nu\psi = a_{\mu\nu}\cdot i\bar\psi\gamma_\nu\psi = a_{\mu\nu}s_\mu$

ie s_μ is a covariant four-vector.

(iii) $M_{\mu\nu} = i\bar\psi\gamma_\mu\gamma_\nu\psi$ $(\mu\neq\nu)$, and $=0$ $(\mu=\nu)$

$M_{\mu\nu}' = i\bar\psi'\gamma_\mu\gamma_\nu\psi' = i\bar\psi S^{-1}\gamma_\mu\gamma_\nu S\psi = i\bar\psi S^{-1}\gamma_\mu S\cdot S^{-1}\gamma_\nu S\psi = i\bar\psi a_{\mu\rho}\gamma_\rho a_{\nu\sigma}\gamma_\sigma\psi$

$= a_{\mu\rho}a_{\nu\sigma}\cdot i\bar\psi\gamma_\rho\gamma_\sigma\psi = a_{\mu\rho}a_{\nu\sigma}M_{\rho\sigma}$

ie $M_{\mu\nu}$ is a second rank tensor. obviously $M_{\mu\nu} = -M_{\nu\mu}$ if $\mu\neq\nu$

ie $M_{\mu\nu}$ is an antisymmetric tensor of second rank (with components $M_{\mu\mu}=0$) ie it has six components and called a six-vector ($M_{12}, M_{23}, M_{31}, M_{14}, M_{24}, M_{34}$)

(iv) $K_\mu = i\bar\psi\gamma_5\gamma_\mu\psi$ is a pseudo-vector ie vector like angular-momentum. For such

Lecture — 12/1/63.

Ex. 4. p. 71 remaining parts:-

(ii) $\mu = \frac{\lambda(1-2\sigma)}{2\sigma} = \frac{3}{2}(K-\lambda) = \frac{E}{2(1+\sigma)} = \frac{3K(1-2\sigma)}{2(1+\sigma)} = \frac{3KE}{9K-E}$

$\mu = \frac{E}{2(1+\sigma)}$ is the standard relation — using $\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}$, $\frac{\lambda(1-2\sigma)}{2\sigma} = \frac{E}{2(1+\sigma)} = \mu$

$\frac{3}{2}(K-\lambda) = \frac{3}{2}\left(\lambda + \frac{2}{3}\mu - \lambda\right) = \mu$

$\frac{3K(1-2\sigma)}{2(1+\sigma)} = \frac{E}{2(1+\sigma)} = \mu$, using $K = \frac{E}{3(1-2\sigma)}$

$\frac{3KE}{9K-E} = \frac{E^2/(1-2\sigma)}{3E/(1-2\sigma) - E} = \frac{E}{3-(1-2\sigma)} = \frac{E}{2(1+\sigma)} = \mu.$

(iii) $\sigma = \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{3K-\lambda} = \frac{E}{2\mu} - 1 = \frac{3K-2\mu}{2(3K+\mu)} = \frac{3K-E}{6K}$

$\sigma = \frac{\lambda}{2(\lambda+\mu)}$ is the standard relation — Also $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is standard

$\frac{\lambda}{3K-\lambda} = \frac{\lambda}{(3\lambda+2\mu)-\lambda} = \frac{\lambda}{2(\lambda+\mu)} = \sigma$

$\frac{E}{2\mu} - 1 = \frac{3\lambda+2\mu}{2(\lambda+\mu)} - 1 = \frac{\lambda}{2(\lambda+\mu)} = \sigma$

$\frac{3K-2\mu}{2(3K+\mu)} = \frac{(3\lambda+2\mu)-2\mu}{2(3\lambda+2\mu+\mu)} = \frac{3\lambda}{6(\lambda+\mu)} = \frac{\lambda}{2(\lambda+\mu)} = \sigma$

$\frac{3K-E}{6K} = \frac{(3\lambda+2\mu) - \mu(3\lambda+2\mu)/(\lambda+\mu)}{2(3\lambda+2\mu)} = \frac{(\lambda+\mu)-\mu}{2(\lambda+\mu)} = \frac{\lambda}{2(\lambda+\mu)} = \sigma$

(iv) $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\sigma)(1-2\sigma)}{\sigma} = \frac{9K(K-\lambda)}{3K-\lambda} = 2\mu(1+\sigma) = \frac{9K\mu}{3K+\mu} = 3K(1-2\sigma)$

$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is a standard relation

Also $\mu = \frac{E}{2(1+\sigma)}$ a standard relation gives $E = 2\mu(1+\sigma)$

& $K = \frac{E}{3(1-2\sigma)}$ " " $E = 3K(1-2\sigma)$.

$E = \frac{\lambda(1+\sigma)(1-2\sigma)}{\sigma}$ also follows from a standard rel^n, $\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}$

$\frac{9K(K-\lambda)}{3K-\lambda} = \frac{3(3\lambda+2\mu)(\lambda+\frac{2}{3}\mu-\lambda)}{(3\lambda+2\mu)-\lambda} = \frac{2\mu(3\lambda+2\mu)}{2(\lambda+\mu)} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = E.$

$\frac{9K\mu}{3K+\mu} = \frac{3(3\lambda+2\mu)\mu}{3\lambda+2\mu+\mu} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = E.$

- Relativistic wave eqns - Klein Gordan eqn * $\left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - M\right)\psi(x_\nu) = 0$. $\left[\frac{1}{M} = \frac{mc}{\hbar}\frac{\hbar}{mc} = \text{compton wave length}\right]$

67th Lecture - 15/2/64 (Special) - Klein-Gordan eqn and its inadequacy, because the wave eqn has only one component, while for electron of spin, we have two component wave eqns $v_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\end{pmatrix}$ & with spin matrices

$\left[\sigma_x\sigma_y = -\sigma_y\sigma_y = i\sigma_z \quad \sigma_x^2 = \cdot = 1\right]$ $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ [ie M_x, M_y, M_z divides by $\frac{1}{2}\hbar$] - Dirac eqn

$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + M\right)\psi(x_\nu)$ with $\psi(x_\nu)$ a 1-column matrix with four elements & γ_μ being 4×4 matrices (Spinors) - Derivation of Klein-Gordan eqn from the Dirac by operating with $\gamma_\lambda \frac{\partial}{\partial x_\lambda} - M$ requires $\gamma_\lambda\gamma_\mu + \gamma_\mu\gamma_\lambda = 2\delta_{\lambda\mu}$, (since rel^n $E^2 = c^2p^2 + m^2c^4$ should hold) as can be shown by taking the Klein Gordan eqn as $\left(\delta_{\lambda\mu}\frac{\partial^2}{\partial x_\lambda \partial x_\mu} - M^2\right)\psi =$ - 16 basic quantities in small form & using $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ ($\gamma_5^2 = 1$, $\gamma_\lambda\gamma_5 + \gamma_5\gamma_\lambda = 0$), $1, \gamma_\lambda, i\gamma_\lambda\gamma_\mu, i\gamma_5\gamma_\lambda, \gamma_5$ (in latter four square of each basic quantity $= 1$) - Requirement of hermiticity of the γ_λ - Defn by aid of Pauli matrix: $\gamma_K = \begin{pmatrix}0 & -i\sigma_K\\ i\sigma_K & 0\end{pmatrix}$ $(K = 1,2,3)$, $\gamma_4 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ & explicit expressions - Verification of $\gamma_5^2 = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$, $\gamma_\lambda\gamma_\mu + \gamma_\mu\gamma_\lambda = 2\delta_{\lambda\mu}$ by using above expressions - Schiff notation $\alpha_K = i\gamma_4\gamma_K$, $\beta = \gamma_4$ & explicit eqn for time - $\hat{\sigma}_K = \begin{pmatrix}\sigma_K & 0\\ 0 & \sigma_K\end{pmatrix} = i\gamma_4\gamma_5\gamma_K = -i\gamma_{lm}$ (in cyclic order)

68th Lecture - 16/2/65 - $\hat{\sigma}_K$ - (2) particle & current densities, (3) spin of electron [Putting Dirac eqn as $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ with

69th Lecture - $H = ic\gamma_4\gamma_K p_K + mc^2\psi\gamma_4$

69th Lecture - 17/2/65 - Writing $\vec{J} = \vec{M} + \frac{1}{2}\hbar\hat{\vec{\sigma}}$ or $J_K = M_K + \frac{1}{2}\hbar\hat{\sigma}_K$, where $\vec{M} = (\vec{r}\times\vec{p})$ is the orbital angular momentum and showing that $[M_K, H] \neq 0$ for eg $[M_{K=1}, H] = -\hbar c(\vec{r}\times\vec{p})$, and $[\vec{\sigma}_K, H] \neq 0$ by showing that $\left[\vec{\sigma}, H\right] = 2c\gamma_4(\vec{r}\times\vec{p})$ & hence $[J_K, H] = 0$ ie J_K is a constant of the motion ie total A.M = orbital A.M + Spin A.M & latter $= \frac{1}{2}\hbar$ in magnitude ie electron has spin $\frac{1}{2}$ - Magnetic moment of the electron. - Motion of magnetic moment due to orbital motion of electron & electron interacting with a magnetic field, the normal Zeeman effect & associated with $\vec{M}$ (orbital A.M) magnetic moment taken as $-\frac{e}{2mc}\vec{M}$ explains Zeeman effect - Anamolous Zeeman effect necessitate notion of spin vector of A.M $\vec{S}$ and associated with it a ~~magnetic~~ magnetic moment $-\frac{e}{mc}\vec{S}$ (note difference in not having factor $\frac{1}{2}$, as in case of $\vec{M}$ - Derivation of magnetic moment of electron from Dirac eqn //

70th Lecture - 18/2/65 - up to derivation of $\left\{\gamma_\mu\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}A_\mu\right) + M\right\}\psi = 0$ - Showing magnetic moment $= -\frac{1}{2}\hbar\vec{\sigma}\cdot\frac{e}{mc}$ -

71st Lecture - 20/2/65 - Proof of existence of magnetic moment for the electron.

72nd Lecture - 22/2/65 - Rel. invariance of Dirac's equation - To prove that $|a_{\mu\nu}|^2 = 1$, it is perhaps simplest to prove as follows:- $|a_{\mu\nu}|^2 = |a_{\mu\nu}||a_{\mu\nu}| = |a_{\mu\nu}\cdot a_{\mu\nu}| = 1$ [using $|A|\times|B| = |A\times B|$ and $a_{\mu\alpha}a_{\mu\beta} = \delta_{\alpha\beta}$] instead of the complicated method of Mandl used last year - That $a_{44} > 0$ excludes time-reversal is seen by taking the transformations with $x_K' = x_K$ and the inverse transformation $x_K = x_K'$ giving $x_4' = c_{44}x_4$ & $x_4 = c_{44}x'$ ie ~~$t = t'$~~ ie ~~$c_{44} = 1$, $t = t'$ ($c_{44} > 0$)~~ & $c_{44}^2 = 1$ & $c_{44} > 0$ gives $t = t'$ or time inversion or non-invariance only corresponds to $(a_{\mu\nu}) = \begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \mp 1\end{pmatrix}$ - similarly space

(V) $K = \lambda + \frac{2}{3}\mu = \frac{\lambda(1+\sigma)}{3\sigma} = \frac{2\mu(1+\sigma)}{3(1-2\sigma)} = \frac{\mu E}{3(3\mu - E)} = \frac{E}{3(1-2\sigma)}$

$K = \lambda + \frac{2}{3}\mu$ and $K = \frac{E}{3(1-2\sigma)}$ are standard relations

Using exprs for σ and E in terms of λ and μ,

$$\frac{\lambda(1+\sigma)}{3\sigma} = \frac{\lambda\left\{1 + \lambda/2(\lambda+\mu)\right\}}{3\lambda/2(\lambda+\mu)} = \frac{\lambda(3\lambda+2\mu)}{3\lambda} = \lambda + \frac{2}{3}\mu = K$$

$$\frac{2\mu(1+\sigma)}{3(1-2\sigma)} = \frac{2\mu\left\{1 + \lambda/2(\lambda+\mu)\right\}}{3\left\{1 - 2\lambda/2(\lambda+\mu)\right\}} = \frac{2\mu(3\lambda+2\mu)}{3\cdot 2\mu} = \lambda + \frac{2}{3}\mu = K.$$

$$\frac{\mu E}{3(3\mu - E)} = \frac{\mu^2(3\lambda+2\mu)/(\lambda+\mu)}{3\left\{3\mu - \mu(3\lambda+2\mu)/(\lambda+\mu)\right\}} = \frac{\mu^2(3\lambda+2\mu)}{3\mu\cdot 3\mu^2} = \lambda + \frac{2}{3}\mu = K.$$

— Eqm eqns for isotropic elastic solid are (1) & (2) subject to (3)

(1) $\tau_{ij,j} = -F_i$

(2) $\tau_{ij} = \lambda\delta_{ij}\vartheta + 2\mu e_{ij}$

(3) ~~(3)~~(a) $\tau_{ij}\nu_j = \bar{T}_i$ — Boundary condn

(4) $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

(5) $e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$ — compatibility eqns

(1) & (2) constitute 9 eqns for the nine quantities τ_{ij}, u_i to be found subject to (3) ie given F_i & $\bar{T}_i$ — Statement of the two problems — Problem 1: To find τ_{ij}, u_i in interior given F_i and $\bar{T}_i$ — Problem 2: To find τ_{ij}, u_i in interior given F_i and u_i on surface.

Problem 2: leads to Navier's eqn $\mu\nabla^2 u_i + (\lambda+\mu)\frac{\partial\vartheta}{\partial x_i} + F_i = 0$

to be solved subject to boundary condns $u_i = f_i(x_1, x_2, x_3)$

$u_i \to e_{ij} \to \tau_{ij}$ — Here no need to use compatibility condns since we go to e_{ij} from u_i

Problem 1: Here from (3) τ_{ij} is given on boundary from $\bar{T}_i$, and we have to find τ_{ij} in interior & also u_i. To find latter we have to find e_{ij} from the relation reciprocal to (2) ie

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\Theta \quad \ldots (2')$$

& then find u_i from (4) & hence the compatibility condns are to be used.

What is done is to express the compatibility condns (5) in terms of τ_{ij} using (2') and also F_i using (1) also, and then take the resulting compatibility eqns and (1) as the reqd eqns for equilibrium. The resulting compatibility eqns are called the Beltrami-Michell compatibility equations.

Derivation of the Beltrami-Michell compatibility conditions: Using (2') these conditions in e_{ij}

since j cannot be negative. For each j, there are $2j+1$ states. Hence the order of the 2^{nd} rep^n is

no. of rows or columns $= \sum_{j_1-j_2}^{j_1+j_2} (2j+1)$, no. of terms $= (j_1+j_2)-(j_1-j_2)+1 = 2j_2+1$

$$\therefore \sum_{j_1-j_2}^{j_1+j_2} 2j + (2j_2+1) = \{2(j_1-j_2)+2(j_1-j_2+1)+\cdots \text{ to } 2j_2+1 \text{ terms}\} + (2j_2+1)$$

$$= 2(j_1-j_2)(2j_2+1) + 2(1+2+\cdots \text{ to } 2j_2 \text{ term}) + (2j_2+1)$$

$$= 2(j_1-j_2)(2j_2+1) + 2\cdot\tfrac{1}{2}\cdot 2j_2(2j_2+1) + (2j_2+1)$$

$$= (2j_2+1)\{2(j_1-j_2)+2j_2+1\} = (2j_2+1)(2j_1+1).$$ just as in C(i).

For the particular case of the combination of orbital A.M, $j_1=1$ and spin A.M, $j_2=\frac{1}{2}$, C(ii) is equivalent to expressing $\Phi_{3/2,3/2}$, $\Phi_{3/2,\frac{1}{2}}$, $\Phi_{3/2,-\frac{1}{2}}$, $\Phi_{3/2,-3/2}$, $\Phi_{\frac{1}{2},\frac{1}{2}}$, $\Phi_{\frac{1}{2},-\frac{1}{2}}$ * as linear combinations of the six products of ψ_{11}, ψ_{10}, $\psi_{1,-1}$, $\psi_{1/2}$, $\psi_{-1/2}$ two each. → use in theory of atomic spectra [* $j_1+j_2 = 3/2$, $j_1+j_2-1 = \frac{1}{2} = j_1-j_2$ Hence $\Phi_{3/2}$ & $\Phi_{1/2}$ with comp. m's]

65th Lecture – 9/2/65 – Remaining portion of previous lecture introducing the ~~notation~~ table from Feenberg & Pake, p. 23 viz

m_1	m_2	$m = m_1+m_2$	j
j_1	j_2	j_1+j_2	j_1+j_2
j_1	j_2-1	j_1+j_2-1	j_1+j_2-1
j_1-1	j_2	"	"
j_1	j_2-2	j_1+j_2-2	j_1+j_2-2
j_1-1	j_2-1	"	"
j_1-2	j_2	"	"
—	—	—	—
—	—	—	—
—	—	—	—
j_1	$-j_2$	j_1-j_2	j_1-j_2
j_1-1	$-j_2+1$	"	"
j_1-2	$-j_2+2$	"	"
⋮	⋮	"	"
⋮	⋮	"	"
⋮	⋮	"	"
j_1-2j_2	j_2	"	"

(in the several blocks the smaller j_2 has values $j_2, j_2-1, j_2-2, \ldots\ldots -j_2$)

which is also in accord with the remark on p. 58 of Condon & Shortley viz "the least value of j_1+j_2-n is obtained when $j_\delta - n = -j_\delta$ where j_δ is the smaller of j_1 and j_2, and this gives $n = 2j_\delta$, and the least value of $j = j_1-j_2$. – $\sum_{j_1-j_2}^{j_1+j_2}(2j+1) = (2j_1+1)(2j_2+1)$ – System of identical particles, symmetry & antisymmetry; formation of these from unsymmetrised functions, exchange degeneracy when H does not involve the spins; for $n=2$, $u(1,2)$ & $u(2,1)$ can be formed from the two functions

$u(1,2) \pm u(2,1)$, but $n=3$, from the two sym. & antisym. fns. viz

$u(1,2,3)+u(2,3,1)+u(3,1,2) \pm u(2,1,3)+u(3,2,1)+u(1,3,2)$

cannot be formed all the six $u(1,2,3)$, etc, but only two linear combinations, the remaining four linear combinations are not of any physical significance / – Pauli's Exclusion Principle [$H_0 = H_0'(1)+H_0'(2)+\cdots+H_0'(n)$, $H_0 = H_0(1,2\ldots n)$

$u(1,2,\ldots n) = v_1(1)v_2(2)\ldots v_n(n)$, $E = E_1+E_2+\cdots+E_n$

$H_0'(1)v_i(1) = E_i v_i(1)$, etc.

$$u_A = \begin{vmatrix} v_1(1), & v_2(1), & \cdots & v_n(1) \\ v_1(2), & v_2(2), & \cdots & v_n(2) \\ \cdots & \cdots & \cdots & \cdots \\ v_1(n), & v_2(n), & \cdots & v_n(n) \end{vmatrix}$$

66th Lecture – 12/2/65 – Pauli's Exclusion Principle – Connection with statistical mechanics, Bose–Einstein and Fermi–Dirac statistics (bosons & fermions). (π-mesons, photons) (electrons, etc. – neutrinos]

Statistics – Connection between spin & statistics [~~Considering only the spin A.M, denoted by s instead of j, of a number an aggregate of fundamental particles~~ [Total spin of a system of identical particles can be an integer or half-odd integer, and experimental evidence shows that ~~the~~ wave must be symmetric or antisymmetric] – Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$.